WDM Optical Switching Networks Using Sparse Crossbars

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Abstract—In this paper, we consider cost-effective designs of wavelength division multiplexing (WDM) optical switching networks for current and future generation communication systems. Based on different target applications: we categorize WDM optical switching networks into two connection models: the wavelength-based model and the fiber-link-based model. Most of existing WDM optical switching networks belong to the first category. In this paper we present new designs for WDM optical switching networks under both models by using sparse crossbar switches instead of full crossbar switches in combination with wavelength converters. The newly designed sparse WDM optical switching networks have minimum hardware cost in terms of both the number of crosspoints and the number of wavelength converters. The single stage and multistage implementations of the sparse WDM optical switching networks are considered. An optimal routing algorithm for the proposed sparse WDM optical switching networks is also presented.

Index Terms—Wavelength division multiplexing (WDM), optical switching networks, optical switches, network architectures, sparse crossbars, concentrators, wavelength conversion, permutation, multicast, multistage networks.

I. INTRODUCTION

Currently, there exists an enormous demand for bandwidth from many emerging networking and computing applications, such as data-browsing in the world wide web, video conferencing, video on demand, E-commerce and image distributing. Optical networking is a promising solution to this problem because of the huge bandwidth of optics. As we know, a single optical fiber can potentially provide a bandwidth of nearly 50 terabits per second, which is about four orders of magnitude higher than electronic data rates of a few gigabits per second accessed by the attached electronic devices such as network processors or gateways. Wavelength division multiplexing (WDM) is a promising technique to exploit such a huge opto-electronic bandwidth mismatch. It divides the bandwidth of an optical fiber into multiple wavelength channels so that multiple devices can transmit on distinct wavelengths through the same fiber concurrently. There has been a lot of research work on WDM optical networks in the literature over the past few years, see, for example, [1]-[12].

A WDM optical switching network provides interconnections between a group of input fiber links and a group of output fiber links with each fiber link carrying multiple wavelength channels. It not only can provide much more connections than a traditional electronic switching network, but also can offer much richer communication patterns for various networking applications. Such an optical switching network can be used to serve as an optical crossconnect (OXC) in a wide-area communication network or to provide high-speed interconnections among a group of processors in a parallel and distributed computing system. A challenge is how to design a high performance WDM optical switching network with low hardware cost. As will be seen later, a cost-effective design of WDM optical switching networks requires non-trivial extensions from their electronic counterpart.

Another challenge in designing WDM optical switching networks is how to keep data in optical domain so as to eliminate the need for costly conversions between optical and electronic signals (so-called O/E/O conversions). To meet the challenge, it is required that either the wavelength on which the data is sent and received has to be the same, or an all-optical wavelength converter needs to be used to convert the signals on an input wavelength to an output wavelength. Thus, in designing a cost-effective WDM optical switching network, we need to reduce not only the number of crosspoints of the switching network but also the cost of wavelength converters. We often have to make trade-offs between the connecting capability of a WDM optical switching network and the number of wavelength converters required along with other design factors.

In this paper, we propose several efficient designs for WDM optical switching networks. In Section II, we first consider two different connection models for WDM optical switching networks: the wavelength-based model and the fiber-link-based model, and then discuss existing design schemes, which are generally under the wavelength-based model. In Sections III, we present new designs for WDM optical switching networks with minimum cost under the wavelength-based model and the fiber-link-based model by using sparse crossbars. In Section IV, we consider the multistage implementation of the proposed optical switching networks. Section V gives a comparison in hardware cost between the new designs and previous ones. Finally, Section VI concludes the paper, and the Appendix contains some mathematical proofs.

II. BACKGROUND AND PREVIOUS WORK

Based on different applications, WDM optical switching networks can be categorized into two connection models: the wavelength-based model and the fiber-link-based model, depending on whether a single device attached to the switching network occupies a single input/output wavelength or a single input/output fiber link. Under the wavelength-based mod-
el, each device occupies one wavelength on an input/output fiber link of a WDM optical switching network. Under the fiber-link-based model, each device occupies an entire input/output fiber link (with multiple wavelength channels) of a WDM optical switching network. These two models are used in different types of applications. In the former each device could be an independent, simple device that needs only one communication channel, and in the latter each device could be a more sophisticated one with multiple input/output channels, such as a network processor capable of handling concurrent, independent packet flows, for example, MMC Networks’ NP3400 processor [13] and Motorola’s C-port network processor [14]. Also, some “hybrid” models are possible, e.g. adopting the wavelength-based model on the network input side and the fiber-link-based model on the network output side. As can be expected, a switching network with wavelength-based model has stronger connection capabilities than that with fiber-link-based model, but it has higher hardware cost.

In addition, the communication patterns realizable by an optical switching network can be categorized into permutation (one-to-one), multicast (one-to-many) and so on. Apparently, the exact definitions of these terms under different connection models could be somewhat different. In permutation communication under the wavelength-based model, each idle input wavelength can be connected to any idle output wavelength with the restriction that an input wavelength cannot be connected to more than one output wavelengths and no two input wavelengths can be connected to the same output wavelength. In multicast communication under the wavelength-based model, each idle input wavelength can be connected to a set of idle output wavelengths, but no two input wavelengths can be connected to the same output wavelength. For presentation convenience, in the rest of the paper we refer to multiple such multicast connection requests as a multicast assignment, and if every output wavelength is involved in a multicast connection in the assignment, the multicast assignment is called a full multicast assignment.

For communication patterns under the fiber-link-based model, since they were not fully discussed in the literature, we elaborate them more here. The possible connections between a number of input fiber links and a number of output fiber links (with each fiber link carrying $k$ wavelengths) can be illustrated as a bipartite graph as shown in Fig. 1, where each link between two nodes in the bipartite graph is actually a $k$-fold link. The major difference is that the wavelengths on a fiber link are treated as indistinguishable ones in the fiber-link-based model. That is, the connections are between the input and output fiber links, not between the input and output wavelengths as in the wavelength-based model.

In permutation communication under the fiber-link-based model, each input (output) fiber link can be involved in up to $k$ independent one-to-one connections to $k'$ ($1 \leq k' \leq k$) output (input) fiber links. Notice that any pair of idle wavelengths on an input fiber link and an output fiber link can be illustrated as a bipartite graph as shown in Fig. 1, where WDM switching networks are used as crossconnects (nodes) in a wide area network (WAN), and the input (output) fiber links of WDM switching networks are links in the WAN. Thus, two destination wavelengths of a multicast connection being on one fiber link implies that two independent channels on some fiber link in the WAN carry the same message. Clearly, it wastes network bandwidth and violates the principle of multicast communication. In either the wavelength-based model or the fiber-link-based model, when a multicast connection involves more than one destination wavelengths on the same fiber link at some node, the multicast route in any intermediate WDM switching network is still connected to only one of the destination wavelengths, and it is the final destination node’s responsibility to relay the multicast message to the rest of destination wavelengths.

For a $k$-wavelength WDM switching network with $N$ input fiber links and $N$ output fiber links, it is interesting to know how many different permutation and multicast connection patterns the WDM switching network can realize. For such a WDM switching network under the wavelength-based model, the number of permutation assignments realizable can be easily calculated as

$$NW_{\text{perm}} = (Nk)!$$

(1)

and the number of multicast assignments realizable is

$$NW_{\text{mcast}} = \left[ {Nk \choose k} k! \right]^N$$

(2)

according to [12]. However, it is much more difficult to calculate the numbers of permutation and multicast assign-
ments for a WDM switching network under the fiber-link-based model. Here we give some bounds on such numbers.

Lemma 1: For a k-wavelength WDM switching network with N input fiber links and N output fiber links under the fiber-link-based model, let the numbers of different full permutation and multicast assignments that can be realized in the network be \( N_{\text{F,perm}} \) and \( N_{\text{F,mcast}} \), respectively. We have that

\[
\prod_{t=1}^{N-k} (N-t+1)! \leq N_{\text{F,perm}} \leq \left( \frac{(Nk)!}{(k)^N} \right),
\]

(3)

where \( s(k, N) = \min\{\frac{\sqrt{8k+1}-1}{2}, N\} \), and

\[
\left( \frac{N^k}{k} \right) \leq N_{\text{F,mcast}} \leq \left( \frac{N^k}{k} \right)^N
\]

(4)

Proof. See Appendix.

Finally, with respect to nonblocking capability, WDM switching networks can be categorized into strictly nonblocking, wide-sense nonblocking, and rearrangeably nonblocking (or simply rearrangeable). In a strictly nonblocking switching network, any legal connection request can be arbitrarily realized without any disturbance to the existing connections. Different from a strictly nonblocking network, in a wide-sense nonblocking switching network, a proper routing strategy must be adopted in realizing any connection requests to guarantee the nonblocking capability. In a rearrangeably nonblocking switching network, any legal connection request can be realized by permitting the rearrangement to on-going connections in the network. Rearrangeable switching networks are usually adopted in applications with scheduled, synchronized network connections, in which case, rearrangement to on-going connections could be avoided.

From the above discussions, we can see that a WDM optical switching network does offer much richer communication patterns than a traditional electronic switching network. For example, in a permutation under the wavelength-based model, a specific wavelength on the input side can be connected only to a specific wavelength on the output side, while in a permutation under the fiber-link-based model, a wavelength on a specific input fiber link can be connected to any one of the wavelengths on a specific output fiber link. As will be seen later, this difference in connection models will lead to switching network designs with different costs.

There has been a considerable amount of work in the literature, e.g. [3], [9], [10], [11], on the wavelength requirement in a WDM network to support permutation and/or multicast communication patterns among the nodes of the network. We view this type of work is in the category of the fiber-link-based model, because they actually pursue that for a given network topology (with fixed parameters), how many wavelengths are required in the network so that the network can realize all permutation (or multicast) connections among the network nodes. On the other hand, under the wavelength-based model, it pursues that for a given network topology and the number of wavelengths per fiber link, under what network parameters we can achieve permutation (or multicast) between input wavelengths and output wavelengths with a certain type of nonblocking capability.

In this paper, we propose optimal designs of WDM optical switching networks under both the wavelength-based and fiber-link-based models for various communication patterns. In the following, if not specifically mentioned, the WDM optical switching network we consider is under the wavelength-based model.

In general, the switching network considered in this paper has N input fiber links and N output fiber links, with each single fiber link carrying k wavelengths \( \lambda_1, \lambda_2, \ldots, \lambda_k \). The set of input links is denoted as \( I = \{i_1, i_2, \ldots, i_N\} \) and the set of output links is denoted as \( O = \{o_1, o_2, \ldots, o_N\} \). An input wavelength \( \lambda_{i_j} \) on link \( i_j \) is denoted as \( (i_j, \lambda_{i_j}) \) and an output wavelength \( \lambda_{o_p} \) on link \( o_p \) is denoted as \( (o_p, \lambda_{o_p}) \). An input wavelength can be connected to an output wavelength through the switching network according to certain communication patterns.

A typical WDM optical switching network consists of demultiplexers, multiplexers, splitters, combiners, and wavelength converters. The demultiplexers are used to decompose input fiber links to individual wavelength signals, the multiplexers are used to combine individual wavelength signals to output fiber links, splitters and combiners perform cross-connecting functions among wavelength signals, and wavelength converters are used to change the wavelengths of signals. Semiconductor optical amplifiers (SOAs) are also used to pass or block selected signals. Fig. 3 gives an example of such a switching fabric. An output of a splitter and an input of a combiner contribute one crosspoint of the optical switching network. A major design issue is to find the minimal possible number of crosspoints for such a switching network.

For an \( N \times N \) WDM optical switching network with \( k \)
wavelengths, we can adopt different design schemes. In some existing designs, e.g., [1], [7], [12], the network can be decomposed into $k \times N \times N$ crossconnects as shown in Fig. 4(a), where connections in the $i^{th} N \times N$ crossconnect are all on wavelength $\lambda_i$. This design scheme has the lowest number of crosspoints compared to other schemes. However, it is only suitable for communication patterns in which the same wavelength is assigned to the source and destination of a connection. For example, it cannot realize one-to-one connections $(i_1, \lambda_1) \rightarrow (o_1, \lambda_2)$, $(i_2, \lambda_2) \rightarrow (o_1, \lambda_1)$ and $(i_2, \lambda_1) \rightarrow (o_1, \lambda_3)$.

One may argue that the design can be improved by adding a set of wavelength converters between the outputs of all $N \times N$ 1-wavelength crossconnects and the output fiber links as shown in Fig. 4(b). Certainly, it can realize more communication patterns, for example, one-to-one connections $(i_1, \lambda_1) \rightarrow (o_1, \lambda_2)$ and $(i_2, \lambda_2) \rightarrow (o_1, \lambda_1)$ now are realizable. However, this is not sufficient for realizing all such communication patterns. For example, it cannot realize an additional legal connection $(i_2, \lambda_1) \rightarrow (o_1, \lambda_3)$ because the $N \times N$ crossconnect with wavelength $\lambda_1$ has only one output to the first output fiber link.

On the other hand, one could consider the scheme that an $N \times N$ WDM optical switching network with $k$-wavelengths is equivalent to an $Nk \times Nk$ crossconnect followed by $Nk$ wavelength converters as shown in Fig. 5. Clearly, an arbitrary permutation can be realized in a permutation WDM optical switching network adopting this design scheme. In the existing designs, an $Nk \times Nk$ crossconnect consists of one stage or multistage full crossbar(s). However, as will be seen in the next section, these designs do not always yield the minimum number of crosspoints for switching networks under different connection models.

### III. New Designs of WDM Switching Networks Using Sparse Crossbars

In our new designs, we still consider the scheme that always places one wavelength converter immediately before each output wavelength shown in Fig. 5. Different from the existing designs, sparse crossbars instead of full crossbars are used to build an $Nk \times Nk$ crossconnect, so that the number of crosspoints of a WDM optical switching network can be reduced.

The question is whether we can use a sparsely connected $Nk \times Nk$ crossconnect and still guarantee that a WDM optical switching network possesses full connecting capability (e.g. realizing an arbitrary permutation or a multicast assignment). An important fact we may make use of in our design is that the placement of wavelength converters can eliminate the need to distinguish the $k$ outputs on a single output fiber link of a switching network. In other words, we can consider the $k$ wavelengths on an output fiber link as a group and do not distinguish their order within the group. We will formally prove the correctness of the WDM switching network designs based this concept later in this section.

In this paper, we consider using a type of sparse cross-
bars, concentrators (as defined below), to design WDM optical switching networks with optimal hardware cost.

A. Concentrators and Reverse Concentrators

In general, a \( p \times q \) \((p \geq q)\) concentrator is a sparse crossbar with \( p \) inputs and \( q \) outputs, in which any \( q \) inputs can be connected to the \( q \) outputs without distinguishing their order. There has been a lot of work on concentrators, see, for example, [18]-[22]. In [20], a lower bound on the number of crosspoints for a \( p \times q \) concentrator was given to be \((p - q + 1)q\). In the literature, some \( p \times q \) concentrators with the minimum \((p - q + 1)q\) crosspoints were designed, such as the so-called fat-and-slim concentrator in [21] and banded concentrator in [22]. In these designs, each output link of the concentrator has a degree of \((p - q + 1)\). Clearly, the number of crosspoints in designs [21], [22] matches the lower bound and thus the designs are optimal. Also, notice that the number of crosspoints is much less than the \( p \cdot q \) crosspoints of a \( p \times q \) full crossbar. In this paper, we will adopt the banded concentrator which has a more regular crosspoint layout.

The \( p \times q \) (banded) concentrator in [22] can be described as a banded sparse crossbar. That is, each of the consecutive \( p - q + 1 \) inputs \( i, i + 1, \ldots, p - q + i \) has a crosspoint to output \( i \), for \( 1 \leq i \leq q \). It was indirectly proved in [22] that a \( p \times q \) sparse crossbar described above is a concentrator by showing its equivalence to a fat-and-slim concentrator. In this paper, we give a direct proof for the following theorem to further demonstrate its concentration capability. Our direct proof also implicitly provides a routing algorithm for banded concentrators.

Theorem 1: A \( p \times q \) \((p \geq q)\) banded sparse crossbar described above is a concentrator (and thus called a banded concentrator).

Proof. See Appendix.

Fig. 6(a) and (b) show a \( 6 \times 3 \) concentrator and its crosspoint layout. As can be seen, the number of crosspoints in a \( 6 \times 3 \) concentrator is 12, which is less than 18, the number of crosspoints in a \( 6 \times 3 \) full crossbar. Also, from the crosspoint layout, it can be verified that any three inputs can be connected to the three outputs.

In this paper, we introduce reverse concentrators which will also be used in the designs of WDM optical switching networks. A \( q \times p \) \((p \geq q)\) reverse concentrator is a sparse crossbar with \( q \) inputs and \( p \) outputs, in which any \( q \) of \( p \) outputs can be connected to the \( q \) inputs without distinguishing their order. We still consider the banded reverse concentrator. Its definition is symmetric to that of a banded concentrator. That is, each of the consecutive \( p - q + 1 \) inputs \( i, i + 1, \ldots, p - q + i \) has a crosspoint to input \( i \), for \( 1 \leq i \leq q \).

Fig. 6(c) and (d) show a \( 3 \times 6 \) reverse concentrator and its crosspoint layout. It can be verified that any three outputs can be connected to the three inputs.

B. Sparse WDM Switching Networks Using Concentrators

B.1 Construction of Sparse WDM Switching Networks

We now consider using concentrators in a single stage WDM optical switching network to reduce the network cost. Since in an \( Nk \times Nk \) crossconnect, every \( k \) outputs corresponding to \( k \) wavelengths of an output fiber link may be indistinguishable in routing, we can use an \( Nk \times Nk \) (banded) concentrator to connect all \( Nk \) inputs and the \( k \) outputs as shown in Fig. 7(a). Thus, for \( N \) output fiber links, we use \( N \) such concentrators to connect all the \( Nk \) inputs and all the \( Nk \) outputs as shown in Fig. 7(b) so that every \( k \) outputs are indistinguishable. Such an \( Nk \times Nk \) crossconnect is simply called output-indistinguishable sparse crossbar.

Similarly, we can use reverse concentrators to construct an \( Nk \times Nk \) crossconnect to connect all \( Nk \) inputs and \( Nk \) outputs so that every \( k \) inputs are indistinguishable. This type of crossconnect is called input-indistinguishable sparse crossbar. The construction is to put \( N \) \( k \times Nk \) reverse concentrators together by sharing the \( Nk \) outputs and can be viewed as flipping the crossconnect in Fig. 7(b) between its inputs and outputs.

We are interested in whether there exists a crossconnect that can function as both an output-indistinguishable sparse crossbar and an input-indistinguishable sparse crossbar, and if it exists, what its cost would be. Such a crossconnect is called bi-directional-indistinguishable sparse crossbar.
shown in Fig. 8(a). The answers for these questions are positive, and we can have the following construction for this type of crossconnect.

The crosspoint layout for a concentrator (Fig. 6(b)) or a reverse concentrator (Fig. 6(d)) can be expressed as a zero-one matrix with entries 0 representing no crosspoint and 1 representing a crosspoint in the position for the corresponding input/output pairs. Moreover, an $Nk \times k$ banded concentrator or a $k \times Nk$ reverse banded concentrator can be expressed as a block matrix consisting of three types of $k \times k$ sub-matrices: full, upper-triangle, and lower-triangle matrices. Also notice that swapping between the rows of the block matrix for a concentrator or swapping between the column- s of the block matrix for a reverse concentrator yield an equivalent concentrator or a reverse concentrator, respectively. Clearly, an $N \times 1$ (or $1 \times N$) block matrix for an $Nk \times k$ concentrator (or a $k \times Nk$ reverse concentrator) consists of an upper-triangle and a lower-triangle, with the rest being full $k \times k$ matrices.

Now we construct an $Nk \times Nk$ bi-directional-indistinguishable sparse crossbar as an $N \times N$ block matrix $M = (M_{i,j})$ such that each of its columns represents an $Nk \times k$ concentrator and each of its rows represents a $k \times Nk$ reverse concentrator. The construction for the matrix $M$ is as follows: $M_{i,j}$ is a lower-triangle sub-matrix for $1 \leq i = j \leq N$; $M_{i,j}$ is an upper-triangle sub-matrix for $(1 \leq i \leq N - 1 \& j = i + 1)$ and $(i = N \& j = 1)$; and $M_{i,j}$ is a full sub-matrix for the rest of $(i,j)$ entries. Fig. 8(b) shows the block matrix for $N = 4$.

It can be easily verified that such a sparse crossbar is both input-indistinguishable and output-indistinguishable. Furthermore, the bi-directional-indistinguishable sparse crossbar has the same cost as the sparse crossbar shown in Fig. 7(b) and the reverse one. Also notice that the new sparse crossbar construction is more balanced in terms of the traffic between inputs and outputs. In the rest of this paper, a sparse crossbar always means a bi-directional-indistinguishable sparse crossbar.

Finally, we can obtain a sparse $N \times N k$-wavelength WDM optical switching network as follows. The network is constructed as in Fig. 5 with the $Nk \times Nk$ crossconnect replaced by the sparse crossbar constructed in Fig. 8(a). Since this sparse crossbar is both input-indistinguishable and output-indistinguishable, it makes no difference for the construction of an $N \times N k$-wavelength WDM optical switching network using a single stage sparse crossbar under the wavelength-based model and under the fiber-link-based model. However, it does make differences when using a multistage crossconnect as discussed in Section IV.

B.2 Connection Capabilities of the Sparse WDM Switching Networks

In the following, we show that the sparse WDM switching network constructed by the concentrators under both the wavelength-based model and the fiber-link-based model has strong connection capabilities.

**Theorem 2:** The sparse WDM switching network under the wavelength-based model has full permutation capability for all input/output wavelengths.

**Proof.** It can be seen from Theorem 1 and the definition of a concentrator that for $k$ outputs (corresponding to an output fiber link) of an $Nk \times Nk$ concentrator, any $k$ inputs among the $Nk$ inputs of the crossconnect can reach the $k$ outputs without distinguishing their order. Also, for a full permutation which maps $Nk$ input wavelengths to $Nk$ output wavelengths, the $k$ input wavelengths mapped to the $k$ output wavelengths corresponding to one output fiber link do not have any conflicts with other input and output wavelength mappings in the permutation. In other words, for a permutation, mappings in different concentrators are independent. Thus, combined with the function of wavelength converters on the output side, the $N \times N$ WDM optical switching network has full permutation capability for all input/output wavelengths. For example, assume that input wavelength $(i_1, \lambda_{e_1})$ is connected to output wavelength $(o_2, \lambda_{e_2})$. In the $j^{th}$ $Nk \times k$ concentrator of the crossconnect, $(i_1, \lambda_{e_1})$ is routed to some (say, the $k^{th}$) output of the concentrator. Finally, the wavelength converter attached to the $k^{th}$ output of the $j^{th}$ concentrator converts the signal to wavelength $\lambda_{e_2}$. ■

**Theorem 3:** The sparse WDM switching network under the wavelength-based model has full multicast capability for all input/output wavelengths.
The proof of Theorem 1 implicitly provides such a route-
capabilities of the constructed sparse switching network un-
to one output fiber link are involved in different 
congressibly to different input wavelengths. Therefore, the multicast assign-
ent can be performed by the N concentrators in the 
cross-connect independently, and finally converted to pre-specified 
wavelengths through the wavelength converters on the output 
side.

We also have the following conclusion for the connection 
capabilities of the constructed sparse switching network under 
the fiber-link-based model.

**Theorem 4:** The sparse WDM switching network under 
the fiber-link-based model has full permutation and multicast 
capabilities for all input/output wavelengths.

B.3 Routing Algorithm in the Sparse WDM Switching Network

As in the proofs of Theorems 2-4, permutation routing and 
multicast routing in the sparse WDM switching network rely 
on a routing algorithm for each individual concentrator. The 
proof of Theorem 1 implicitly provides such a routing 
algorithm for banded concentrators. Since the proof in- 
volves P. Hall’s Theorem on a system of distinct representa-
tives, the routing algorithm for a typical Nk × k concentrator 
can adopts M. Hall’s algorithm [23], which yields O((Nk)^2) 
time complexity. Fortunately, by taking advantage of the reg- 
ular structure of the banded concentrator, we can have a much 
faster routing algorithm for the concentrator only in O(k) 
time.

The algorithm concentrator-routing() for a p × q (p ≥ 2q) 
concentrator shown in Table 1 takes any of its q inputs, and 
makes a mapping to the q outputs. Recall that from the proof 
of Theorem 1, all the p inputs can be divided into three seg-
ments A, B, and C. Among them, A and C correspond to 
the lower-triangle and upper-triangle q × q zero-one matrices, 
respectively. In Step 1, the q inputs are partitioned to three 
parts as in segments A, B, and C, and the elements in 
subsets of A and C are sorted. In Step 2, the global vari-
ables leftbound and rightbound, indicating the boundaries of 
mapped outputs from the left side (smaller labels) and the 
right side (larger labels) respectively, are initialized. In Steps 
3 and 4, for inputs in segment A, an input with a smaller 
label has been mapped to an output with a smaller label from 
the left side; and for inputs in segment C, an input with a 
larger label has been mapped to an output with a larger label 
from the right side. In Step 5, the inputs in segment B are 
mapped to the outputs between leftbound and rightbound. 
From the construction of a banded concentrator, we can see 
that this algorithm maps any q inputs to the q outputs without 
any conflict.

<table>
<thead>
<tr>
<th><strong>TABLE 1</strong></th>
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<tbody>
<tr>
<td><strong>ROUTING ALGORITHM FOR A p × q CONCENTRATOR</strong></td>
</tr>
<tr>
<td><strong>Input:</strong> i1, i2, ..., iq; // q inputs of the concentrator</td>
</tr>
<tr>
<td><strong>Output:</strong> mapping[1..q]; // map each output o_j to some input i_j</td>
</tr>
<tr>
<td><strong>Step 1:</strong> let the q inputs be divided by input segments A, B, C:</td>
</tr>
<tr>
<td>i_{a1}, i_{a2}, ..., i_{aq}, j_{b1}, j_{b2}, ..., j_{bq}, i_{c1}, i_{c2}, ..., i_{cq};</td>
</tr>
<tr>
<td>where q_1 + q_2 + q_3 = q with q_1, q_2, q_3 ≥ 0;</td>
</tr>
<tr>
<td>Suppose i_{a1} ≤ i_{aq} &lt; i_{aq+1} and i_{c1} ≤ i_{cq} &lt; i_{cq+1};</td>
</tr>
<tr>
<td><strong>Step 2:</strong> leftbound = 1; rightbound = q;</td>
</tr>
<tr>
<td><strong>Step 3:</strong> for (j = 1; j &lt; q_j; j++) {</td>
</tr>
<tr>
<td>s = leftbound++;</td>
</tr>
<tr>
<td>mapping[s] = i_{aj}; // map o_s to i_{aj};</td>
</tr>
<tr>
<td>}</td>
</tr>
<tr>
<td><strong>Step 4:</strong> for (j = q_j; j ≥ 1; j--) {</td>
</tr>
<tr>
<td>s = rightbound--;</td>
</tr>
<tr>
<td>mapping[s] = i_{cj}; // map o_s to i_{cj};</td>
</tr>
<tr>
<td>}</td>
</tr>
<tr>
<td><strong>Step 5:</strong> for (j = 1; j &lt; q_j; j++) {</td>
</tr>
<tr>
<td>s = leftbound++;</td>
</tr>
<tr>
<td>mapping[s] = i_{bj}; // map o_s to i_{bj};</td>
</tr>
<tr>
<td>}</td>
</tr>
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</table>

For the time complexity of the algorithm, we can see that 
it takes O(q) for Steps 3 to 5. For the initialization in Step 
1, since the label of an input can determine which segment 
it belongs to, it takes O(q) time to do the partition of the q 
inputs. Also, since the lengths of segments A and C are both 
q, we can apply the bucket sorting algorithm to sort elements 
in the subsets of A and C in Step 1, and thus it still takes O(q) 
time. Overall, the time complexity of the algorithm is O(q).

When applying the algorithm to an Nk × k concentrator 
in the sparse WDM switching network, it will take O(k) 
time. The permutation or multicast routing in the Nk × Nk 
crossconnect can be reduced to the routing in N individual 
Nk × k concentrators. Therefore, introducing concentrators 
and adopting the concentrator routing algorithm do not in-
crease the routing time complexity for the sparse switching 
network.

This algorithm can also be easily extended to routing in a 
reverse concentrator.

B.4 Hardware Cost of a Single Stage WDM Switching Network

Since the number of crosspoints of a WDM optical switching 
network is simply that of its crossconnect, we can analyze 
the number of crosspoints for the latter. From our construction, 
we can see that the total number of crosspoints of an 
Nk × Nk crossconnect is (Nk − k + 1)Nk, which will be proved (in the following) to be the minimum possible for this 
type of Nk × Nk crossconnect.

**Lemma 2:** The lower bound on the number of crosspoints of an Nk × Nk crossconnect in which every k outputs are indistinguishable is (Nk − k + 1)Nk.

**Proof:** We only need to show that each output of the Nk × Nk crossconnect is reachable from at least Nk − k + 1 inputs so that the lower bound on the number of crosspoints of the Nk × Nk crossconnect is (Nk − k + 1)Nk. Assume it is not true,
i.e. there exists some output, which is only reachable from at most $Nk - k$ inputs. Thus, there exist at least $k$ inputs which cannot reach this output, as well as the group of the $k$ outputs this output is in. This contradicts with the definition of a concentrator that every $k$ outputs can be reached by any $k$ inputs without distinguishing the order.

Finally, we show that the design of WDM optical switching networks in this section is optimal.

**Theorem 5:** A single stage WDM switching network proposed in this paper has the minimum hardware cost in terms of both the number of crosspoints and the number of wavelength converters.

**Proof.** First, since the newly designed $Nk \times Nk$ crossconnect consists of $N Nk \times k$ concentrators and has $(Nk - k + 1)Nk$ crosspoints which match the lower bound required for an $Nk \times Nk$ crossconnect with every $k$ outputs indistinguishable in Lemma 2, the single stage WDM optical switching network proposed in this paper has the minimum number of crosspoints.

Second, since each input wavelength may require to connect to an output with a different wavelength, the full permutation connection capability between $Nk$ input wavelengths and $Nk$ output wavelengths requires at least $Nk$ wavelength converters. The newly designed WDM optical switching network uses exactly $Nk$ wavelength converters, and thus the design has the minimum number of wavelength converters. ■

### B.5 Nonblocking Capabilities

The newly designed WDM optical switching network may have different nonblocking capabilities depending on the network connection and/or application models. If the model requires to set up the connections in terms of output fiber links (especially under the fiber-link-based model), the WDM optical switching network is strictly nonblocking based on the properties of the concentrators. If the model requires that the connection of each pair of input and output wavelengths is set independently, the WDM optical switching network is rearrangeably nonblocking due to the use of concentrators. Fortunately, in the case of rearrangement, only $k$ signals (on the same output fiber link) may be affected.

### IV. WDM Switching Networks Using Multistage Crossconnects

In this section, we extend the WDM optical switching networks to those using multistage crossconnects so that the number of crosspoints can be further reduced. We first consider a three-stage crossconnect for permutations, and then give a description for a general multistage crossconnect.

A three-stage $Nk \times Nk$ crossconnect under the wavelength-based model consists of $r \times m \times n$ crossbars in the first stage, $m \times r \times r$ crossbars in the middle stage, and $r \times m \times n$ output-indistinguishable sparse crossbars in the third stage as shown in Fig. 9. The values of $n$ and $r$ satisfy that $n r = Nk$, and the value of $m$ depends on the type of the overall optical switching network. For a permutation WDM optical switching network, $m \geq n$ [16]; and for a multicast WDM optical switching network, $m \geq 3(n - 1 \log \log r)$ [17].

Finally, the sparse $N \times N k$-wavelength WDM optical switching network under the wavelength-based model is constructed as in Fig. 5 with the $Nk \times Nk$ crossconnect replaced by the crossconnect in Fig. 9. The sparse $N \times N k$-wavelength WDM optical switching network under the fiber-link-based model is constructed as in Fig. 5 with the $Nk \times Nk$ crossconnect replaced by the crossconnect in Fig. 10.
Theorem 6: The $N \times N$ $k$-wavelength WDM optical switching network in Fig. 5 with the $Nk \times Nk$ three-stage crossconnect in Fig. 9 or Fig. 10 has full permutation and multicast capabilities.

Proof. The permutation and multicast capabilities can be easily verified for a WDM optical switching network under wavelength-based model by using Theorem 2, Theorem 3, and [16], [17].

For a WDM optical switching network under the fiber-link-based model, we can perform the routing as follows. First, we assign proper wavelengths to $k$ channels of each input and output fiber links. Then we perform permutation or multicast routing in the three-stage crossconnect under the wavelength-based model, by assuming that the first stage consists of small full crossbars. Finally, we determine the routing in each small sparse crossbar in the first stage by modifying the routing obtained when assuming it as a small full crossbar. Since for every $k$ inputs of such a sparse crossbar, we know the $k$ outputs they are mapped to, we can make the re-routing from the $k$ outputs to the $k$ inputs in the corresponding reverse concentrator. This re-routing is legal, since under the fiber-link-based model we do not distinguish the wavelengths in an input (as well as output) fiber link. It is achievable by using a routing algorithm (in a reverse concentrator), which is symmetric to that in Table 1.

We now calculate the number of crosspoints for such a three-stage crossconnect under the wavelength-based model. Without loss of generality, let $n$ be evenly divisible by $k$. Using a similar argument to that in the last section, an $m \times n$ (with $m \geq n$) sparse crossbar with every $k$ outputs indistinguishable can be constructed and has $(m-k+1)n$ crosspoints. Thus, the number of crosspoints of the overall three-stage WDM optical switching network under the wavelength-based model is

$$r \cdot nm + m \cdot r^2 + r \cdot (m-k+1)n = Nk\left(2m + \frac{m}{n}r - k + 1\right).$$

For easy calculations, let $\frac{m}{n}$ be bounded by $c$. Clearly, for a permutation switching network, $c = 1$; and for a multicast switching network, $c = O(\frac{\log \log N}{\log N})$. After the optimization, the number of crosspoints is bounded by

$$\min\{Nk[c(2n+r) - k+1]\} = \min\{Nk[c(2n+Nk/n) - k+1]\} = c(2Nk)^{\frac{2}{3}} - Nk(k-1).$$

Similarly, the number of crosspoints of the overall three-stage WDM optical switching network under the fiber-link-based model is

$$m \cdot r^2 + 2r \cdot (m-k+1)n = Nk\left(2m + \frac{m}{n}r - 2k + 2\right).$$

After the optimization, it is bounded by

$$c(2Nk)^{\frac{2}{3}} - 2Nk(k-1).$$

In general, a multistage switching network with more than three stages can be recursively constructed by replacing each single stage crossbar and/or sparse crossbar at a stage with a multistage crossconnect of the same size.

### V. Comparisons of Hardware Costs

In this section we compare hardware costs of WDM switching networks of the previous designs [1], [7], [12] and the new designs in this paper under different models. The hardware cost is a combination of the number of crosspoints, the number of wavelength converters, and the number of multiplexers and demultiplexers. The comparison is shown in Table 2.

In the table, we compare the designs for permutation and multicast switching networks under the single stage and three-stage implementations. Since single stage switching networks for permutation and multicast have the same cost, we list only one item for each of single stage designs without distinguishing their communication patterns. For the three-stage implementation, we list the comparison for permutation and multicast separately. For three-stage multicast switching networks, the previous designs [12] in the table are two recursively defined WDM switching networks denoted as Prev1 and Prev2. In the table, WB and FLB indicate the design being under the wavelength-based model and fiber-link-based model, respectively. The previous designs [1], [7], [12] are under the wavelength-based model only. As can be seen in the comparison, the new designs adopting sparse crossbars in this paper have less hardware cost than that of previous designs for either permutation or multicast and with either the single stage or the multistage implementations.

### VI. Conclusions

In this paper, we first categorized WDM optical switching networks into two different connection models based on their target applications: the wavelength-based model and the fiber-link-based model. We then presented new designs for WDM optical switching networks under both the wavelength-based model and the fiber-link-based model by using sparse crossbar switches instead of full crossbar switches in combination with wavelength converters. The sparse switching networks have the minimum hardware cost.
in terms of both the number of crosspoints and the number of wavelength converters. The single stage and multistage implementations of the sparse switching networks are considered. An optimal routing algorithm for the WDM sparse crossbar is also presented in this paper.

APPENDIX

In this appendix, we provide proofs for Lemma 1 and Theorem 1.

Before we prove Lemma 1, we give the following statement for a better understanding of the problem. Since in a WDM switching network under the fiber-link-based model we treat the wavelengths on a fiber link as identical ones, we are only concerned with the number of wavelengths on an input (output) fiber link connected to some output (input) fiber links. Given any $N \times N$ matrix

$$
\begin{bmatrix}
k_{1,1} & k_{1,2} & \cdots & k_{1,N} \\
k_{2,1} & k_{2,2} & \cdots & k_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
k_{N,1} & k_{N,2} & \cdots & k_{N,N}
\end{bmatrix}
$$

(5)

satisfying $\sum_{i=1}^{N} k_{i,j} = k$, $\sum_{j=1}^{N} k_{i,j} = k$, and $k_{i,j} \in \{0, 1, 2, \ldots, k\}$ for $1 \leq i, j \leq N$, it corresponds to a permutation assignment of the WDM switching network, where each row (column) of the matrix represents an input (output) fiber link of the network. In fact, the sum of elements in row $i$ (that is, $\sum_{j=1}^{N} k_{i,j} = k$) is a partition of integer $k$ so that we can use $k_{i,j}$ wavelengths to realize $k_{i,j}$ independent one-to-one connections from input fiber link $i$ to output fiber link $j$ for $j = 1, 2, \ldots, N$; a similar argument applies to column $j$ of the matrix.

Clearly, $N_{F,\text{perm}}$ should be the number of different matrices in form (5). However, we believe that the enumeration for matrices in form (5) is an unsolved open problem. Instead, we provide some lower and upper bounds for $N_{F,\text{perm}}$ and $N_{F,\text{mcast}}$ in this paper.

**Proof of Lemma 1.** We use the numbers of permutations and multicast assignments that can be realized by the WDM switching network under the hybrid connection model (with the wavelength-based model on the input side and the fiber-link-based model on the output side) as the upper bounds for those under the fiber-link-based model.

Notice that there are $(Nk)!$ permutations that can be realized by the WDM switching network under the wavelength-based model as in (1). We immediately have that the number of permutations that can be realized by the network under the hybrid connection model is $\frac{(Nk)!}{(k!)^s}$, since $k$ wavelengths on each of $N$ output fiber links are indistinguishable. Thus, we have $N_{F,\text{perm}} \leq \frac{(Nk)!}{(k!)^s}$. Similarly, since there are $\left(\binom{Nk}{k}\right)^N$ full multicast assignments that can be realized by the WDM switching network under the wavelength-based model as in (2), we have $N_{F,\text{mcast}} \leq \left(\binom{Nk}{k}\right)^N$.

For a lower bound on $N_{F,\text{perm}}$, we consider a partition of $k$ wavelengths on each fiber link into $s \geq 1$ parts of distinct sizes, such that

$$
k = k_1 + k_2 + \cdots + k_s,
$$

where $k_1 > k_2 > \cdots > k_s > 0$ are positive integers. For simplicity, we first assume that $s \leq N$, and we also call the part of size $k_i$ wavelength group $k_i$. Now we make a special permutation (under the fiber-link-based model) that maps wavelength groups of the same size between the input and output fiber links, and if possible we always let $s$ groups in an input fiber link map to $s$ distinct output fiber links, which we refer to as distinct mapping property of an input (output) fiber link in this paper.

Our task is to estimate how many such permutations. First, we map wavelength groups of size $k_1$ between the input and output fiber links, which yields $N!$ different ways. Secondly, we map wavelength groups of size $k_2$ in the order from the first input fiber link to the last input fiber link, and make sure if possible group $k_2$ on an input fiber link will not map to the same output fiber link that group $k_1$ on the same input fiber link maps to. Clearly, there are at least $N-1$ input (output) fiber links satisfying the distinct mapping property so far, and there are at least $(N-1)!$ different ways. We can similarly map the remaining groups $k_i$ for $3 \leq i \leq s$. Fig. 11 gives an example of such mapping.

![Fig. 11. A permutation under the fiber-link-based model maps wavelength groups of same sizes, where $N = 4$, $k = 6$, $k_1 = 3$, $k_2 = 2$, and $k_3 = 1$. There are 3 input (output) fiber links satisfying the distinct mapping property.](image)
where \( s(k, N) = \min \{ \frac{\sqrt{kN} - 1}{2}, N \} \).

For a lower bound on \( N_{F, mcast} \), we consider a simpler configuration that each of \( k \) wavelengths on an output fiber link is connected from a different input fiber link, so that the wavelengths are selected indistinguishably on the input fiber links. Clearly, there are \( N(N - 1) \cdots (N - k + 1) \) ways to map \( k \) wavelengths on an output fiber link. Because the \( k \) wavelengths on the output side are indistinguishable, the number is reduced to \( \frac{N(N - 1) \cdots (N - k + 1)}{k!} = \binom{N}{k} \), and thus we obtain \( N_{F, mcast} \geq \binom{N}{k} \).

**Proof of Theorem 1.** In a \( p \times q \) banded sparse crossbar constructed in Section III-A, for an input \( i \) (\( 1 \leq i \leq p \)), let \( \phi(i) \) be the set of outputs each of which has a crosspoint to input \( i \). According to Hall’s Theorem [23], each of any \( q \) inputs from \( p \) inputs can be connected to a distinct output (through a crosspoint) if and only if for any \( k \) distinct inputs \( i_1, i_2, \ldots, i_k \) that
\[
|\phi(i_1) \cup \phi(i_2) \cup \cdots \cup \phi(i_k)| \geq k.
\] (7)

Let all inputs (outputs) of a banded sparse crossbar be in the vertical (horizontal) direction. Then all the crosspoints are filled by the points with integral coordinates in a parallelogram. There are basically two cases, \( p \geq 2q \) and \( p \leq 2q \), which have different shapes of parallelograms as shown in Fig. 12 (a) and (b), respectively. In each case, we can divide all the \( p \) inputs into three consecutive segments named A, B and C (see below). Also, notice that for any input \( i \), \( \phi(i) \) is a set of consecutive outputs. For simplicity, we write the set of all (consecutive) integers between \( a \) and \( b \) as \( \{a, a+1, \ldots, b-1, b\} \). We can formally define \( \phi(i) \) as follows.

For case \( p \geq 2q \) (Fig. 12(a)) we have,

Input segment A: \( 1 \leq i \leq q \),
\[ \phi(i) = \{1, \ldots, i\}, \text{ with } |\phi(i)| = i \leq q; \]

Input segment B: \( q < i < p - q + 1 \),
\[ \phi(i) = \{1, \ldots, q\}, \text{ with } |\phi(i)| = q; \]

Input segment C: \( p - q + 1 \leq i \leq p \),
\[ \phi(i) = \{i - p + q, \ldots, q\}, \text{ with } |\phi(i)| = p - i + 1 \leq q \] (8)

For case \( p \leq 2q \) (Fig. 12(b)) we have,

Input segment A: \( 1 \leq i \leq p - q + 1 \),
\[ \phi(i) = \{1, \ldots, i\}, \text{ with } |\phi(i)| = i \leq p - q + 1; \]

Input segment B: \( p - q + 1 < i < q \),
\[ \phi(i) = \{i - p + q, \ldots, i\} \text{ with } |\phi(i)| = p - q + 1; \]

Input segment C: \( q \leq i \leq p \),
\[ \phi(i) = \{i - p + q, \ldots, q\}, \text{ with } |\phi(i)| = p - i + 1 \leq q \] (9)

Now we are in the position to prove (7) holds for the two cases separately.

**Case 1** \( p \geq 2q \):

**Subcase 1.1** An input named \( i_B \) from \( i_1, i_2, \ldots, i_k \) falls into segment B:
From (8) or Fig. 12 (a), we must have
\[ |\phi(i_1) \cup \phi(i_2) \cup \cdots \cup \phi(i_k)| = |\phi(i_B)| = q \geq k, \]
that is, (7) holds.

**Subcase 1.2** All inputs \( i_1, \ldots, i_k \) fall into segment A:
Let the maximum-indexed input among them be \( i_A \). From (8) or Fig. 12 (a), we have
\[ |\phi(i_1) \cup \phi(i_2) \cup \cdots \cup \phi(i_k)| = |\phi(i_A)| = i_A; \]
Also, since all \( k \) distinct inputs are in the input section \( \{1, \ldots, i_A\} \), we must have \( i_A \geq k \), and thus (7) holds.

**Subcase 1.3** All the \( k \) inputs fall into segment C:
Let the minimum-indexed input among them be \( i_C \). From (8) or Fig. 12 (a), we have
\[ |\phi(i_1) \cup \phi(i_2) \cup \cdots \cup \phi(i_k)| = |\phi(i_C)| = p - i_C + 1; \]
Also, since all \( k \) distinct inputs are in the input section \( \{i_C, \ldots, p\} \), we must have that the number of integers in the section is \( p - i_C + 1 \geq k \), and thus (7) holds.

**Subcase 1.4** Some of \( k \) inputs fall into segment A and some fall into segment C:
Let the maximum (minimum)-indexed input among those falling into segment A (C) be \( i_A (i_C) \). We have
\[ |\phi(i_1) \cup \phi(i_2) \cup \cdots \cup \phi(i_k)| = |\phi(i_A) \cup \phi(i_C)|; \]
since from (8) or Fig. 12 (a), \( \phi(i_A) \) is a set of consecutive outputs starting from 1 and \( \phi(i_C) \) is a set of consecutive outputs ended at \( q \), if \( \phi(i_A) \cap \phi(i_C) \neq \phi \),
\[ |\phi(i_A) \cup \phi(i_C)| = q \geq k; \]
otherwise \( \phi(i_A) \cap \phi(i_C) = \phi \), which implies
\[ |\phi(i_A) \cup \phi(i_C)| = |\phi(i_A)| + |\phi(i_C)| = i_A + (p - i_C + 1). \]
Let the numbers of the \( k \) inputs falling into segment A and C be \( k_1 \) and \( k_2 \), respectively, where \( k_1 + k_2 = k \). Then from subcases 1.2 and 1.3, we have \( i_A \geq k_1 \) and \( p - i_C + 1 \geq k_2 \), which implies \( i_A + (p - i_C + 1) \geq k_1 + k_2 = k \), and thus (7) holds.

**Case 2** \( p \leq 2q \):

**Subcase 2.1** All inputs \( i_1, i_2, \ldots, i_k \) fall into segment B:
Let the minimum-indexed and the maximum-indexed inputs be $i_B$, and $i_{B'}$, respectively. From (9) or Fig. 12 (b), we have
\[ |\phi(i_1) \cup \phi(i_2) \cup \cdots \cup \phi(i_k)| = |\phi(i_B) \cup \phi(i_{B'})|, \]
and notice that $\phi(i_B) \cup \phi(i_{B'}) = \{i_B, p+q, \ldots, i_{B'}\}$. Thus,
\[ |\phi(i_B) \cup \phi(i_{B'})| = i_{B'} - i_B + p - q + 1 \geq i_{B'} - i_B + 1 \geq k. \]

**Subcase 2.2** All inputs $i_1, i_2, \ldots, i_k$ fall into segment A: Same as Subcase 1.2.

**Subcase 2.3** All inputs $i_1, i_2, \ldots, i_k$ fall into segment C: Same as Subcase 1.3.

**Subcase 2.4** All k inputs fall into both segments A and C: Same as Subcase 1.4.

**Subcase 2.5** All k inputs fall into both segments A and B:
Let the maximum-indexed input among those falling into segment A be $i_A$, and the minimum-indexed and the maximum-indexed inputs among those falling into segment B be $i_B$, and $i_{B'}$, respectively. From (9) or Fig. 12 (b), we have
\[ |\phi(i_A) \cup \phi(i_{B'})| = \phi(i_A) \cup \phi(i_{B'}) = \{i_A, i_{B'}, i_{B'}\} = \{i_A, i_{B'}, i_{B'}\}, \]
where $k_1$ and $k_2$ are the numbers of the $k$ inputs falling into segment A and B, respectively.

**Subcase 2.6** All k inputs fall into both segments B and C: Same as Subcase 2.5.

**Subcase 2.7** All k inputs fall into all segments A, B, and C:
Let the numbers of the $k$ inputs falling into segments A, B, and C be $k_1, k_2$, and $k_3$, respectively where $k_1 + k_2 + k_3 = k$, and let $i_A, i_B, i_{B'}, i_{C'}$ be defined as before. From (9) or Fig. 12 (b), we have
\[ |\phi(i_A) \cup \phi(i_{B'}) \cup \phi(i_{C'})| = |\phi(i_A) \cup \phi(i_{B'}) \cup \phi(i_{C'})| \]
if $\phi(i_A) \cap \phi(i_{B'}) \neq \phi$ and $\phi(i_{B'}) \cup \phi(i_{C'}) \neq \phi$, we have
\[ |\phi(i_A) \cup \phi(i_{B'}) \cup \phi(i_{C'})| = q \geq k; \]
if $\phi(i_A) \cap \phi(i_{B'}) \neq \phi$ and $\phi(i_{B'}) \cup \phi(i_{C'}) = \phi$, from Subcases 2.5 and 2.3 we have
\[ |\phi(i_A) \cup \phi(i_{B'}) \cup \phi(i_{C'})| = |\phi(i_A) \cup \phi(i_{B'}) \cup \phi(i_{C'})| \]
similarly, if $\phi(i_A) \cap \phi(i_{B'}) = \phi$ and $\phi(i_{B'}) \cup \phi(i_{C'}) \neq \phi$, (7) is also true; Finally if $\phi(i_A) \cap \phi(i_{B'}) = \phi$ and $\phi(i_{B'}) \cup \phi(i_{C'}) = \phi$, from Subcases 2.1, 2.2, and 2.3 we have
\[ |\phi(i_A) \cup \phi(i_{B'}) | = |\phi(i_A) \cup \phi(i_{B'}) | \]
Hence, (7) always holds and the $p \times q$ banded sparse crossbar is a concentrator.

**REFERENCES**


